

Spectral Flow and Feigin-Fuks Parameter Space of N=4 Superconformal Algebras

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Abstract

The parameter space of the Feigin-Fuks representations of the N=4 $SU(2)_k$ superconformal algebras is studied from the viewpoint of the spectral flow. The η phase of the spectral flow is nicely incorporated through twisted fermions and the spectral flow resulting from the inner automorphism of the N=4 superconformal algebras is explicitly shown to be operating as identity relations among the generators. Conditions for the unitary representations are also investigated in our Feigin-Fuks parameter space.

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It is well recognized nowadays that the so-called Feigin-Fuks (FF) representations (or the Coulomb-gas representations) [1, 2] are very important and almost inevitably required tools for investigating the representation theories of the conformal and superconformal algebras. By now we have established the FF representations of the superconformal algebras with higher number of supercharges [3, 4, 5, 6], up to N=4 [7, 8, 9].

On the other hand, the spectral flows resulting from the inner automorphisms of the conformal and superconformal algebras with N=2,3 and 4 were first recognized by Schwimmer and Seiberg [10], and their remarkable implications on the representation theories of the algebras have been discussed by many people [11, 12, 13].

In the present paper we shall study the parameter space of the FF representations of the N=4 $SU(2)_k$ superconformal algebras with particular focus on the properties of their spectral flow which is nicely embedded in our FF parameterization [7]. We shall explicitly show how remarkably the spectral flow emerges by use of the identities holding among the FF parameters. Our study not only shows how the spectral flow for the unitary representations [13, 14, 15] of the N=4 $SU(2)_k$ superconformal algebras is operating, but also establishes it to hold explicitly in the nonunitary representations by use of the continuous parameters of our FF representations [7].

The N=4 $SU(2)_k$ superconformal algebras are defined by the form of the operator product expansions (OPE) among operators given by the energy-momentum tensor $L(z)$, the $SU(2)_k$ local nonabelian generators $T^i(z)$, and the iso-doublet and -antidoublet supercharges $G^a(z)$ and $\bar{G}_a(z)$:

$$\begin{aligned} L(z)L(w) &\sim \frac{3k}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}, \\ T^i(z)T^j(w) &\sim \frac{\frac{1}{2}k\eta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}\eta_{kl}T^l(w)}{z-w}, \quad L(z)T^i(w) \sim \frac{T^i(w)}{(z-w)^2} + \frac{\partial_w T^i(w)}{z-w}, \\ T^i(z)G^a(w) &\sim -\frac{\frac{1}{2}(\sigma^i)^a{}_b G^b(w)}{z-w}, \quad T^i(z)\bar{G}_a(w) \sim \frac{\frac{1}{2}\bar{G}_b(w)(\sigma^i)^b{}_a}{z-w}, \end{aligned}$$

$$\begin{aligned}
L(z)G^a(w) &\sim \frac{\frac{3}{2}G^a(w)}{(z-w)^2} + \frac{\partial_w G^a(w)}{z-w}, \quad L(z)\bar{G}_a(w) \sim \frac{\frac{3}{2}\bar{G}_a(w)}{(z-w)^2} + \frac{\partial_w \bar{G}_a(w)}{z-w}, \\
G^a(z)G^b(w) &\sim 0, \quad \bar{G}_a(z)\bar{G}_b(w) \sim 0, \\
G^a(z)\bar{G}_b(w) &\sim \frac{4k\delta^a_b}{(z-w)^3} - \frac{4(\sigma^i)^a_b \eta_{ij} T^j(w)}{(z-w)^2} - \frac{2(\sigma^i)^a_b \eta_{ij} \partial_w T^j(w)}{z-w} + \frac{2\delta^a_b L(w)}{z-w} \quad (1)
\end{aligned}$$

where $i = (\pm, 0)$ denote SU(2) triplets in the diagonal basis, while the superscripts (subscripts) $a = (1, 2)$ label SU(2) doublet (antidoublet) representations. The symmetric tensors $\eta^{ij} = \eta^{ji} = \eta_{ij}$ are defined in the diagonal basis as $\eta^{+-} = \eta^{00} = 1$, while the antisymmetric tensors $\epsilon^{ijk} = -\epsilon^{jik} = -\epsilon^{ikj}$ are similarly defined as $\epsilon^{+-0} = -i$, etc., and otherwise zero. The Pauli matrices are given by $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/\sqrt{2}$, $\sigma^0 = \sigma^3$. The corresponding notations in terms of the isospin raising and lowering by a half unit for the fermionic operators are given by $G^1 \equiv G^-$, $G^2 \equiv G^+$, $\bar{G}_1 \equiv \bar{G}_+$, $\bar{G}_2 \equiv \bar{G}_-$. The symmetric delta function $\delta^a_b = \delta^b_a$ has the standard meaning with $\delta^{-+} = \delta^1_1 = 1$, $\delta^{++} = \delta^2_1 = 0$, etc..

Here is a remark in order. The $N = 4$ algebras Eq.(1) have a global SU(2) symmetry. As a result, one can consider the infinite number of independent $N = 4$ twisted algebras labeled by ρ corresponding to the conjugate classes of the global automorphism[10]. The FF representations of the ρ -extended $N = 4$ algebras are discussed in a separate paper[16]. The present paper will be restricted to the case of $\rho = 0$.

As noted by Schwimmer and Seiberg [10], the SU(2) gauge symmetry gives the *inner* automorphism

$$\begin{aligned}
L(z) &\rightarrow L(z) + i \frac{d\alpha(z)}{dz} T^0(z) - \frac{k}{4} \left(\frac{d\alpha(z)}{dz} \right)^2, \\
T^0(z) &\rightarrow T^0(z) + i \frac{k}{2} \frac{d\alpha(z)}{dz}, \quad T^\pm(z) \rightarrow e^{\pm i\alpha(z)} T^\pm(z), \\
G^\mp(z) &\rightarrow e^{\mp i \frac{\alpha(z)}{2}} G^\mp(z), \quad \bar{G}_\pm(z) \rightarrow e^{\pm i \frac{\alpha(z)}{2}} \bar{G}_\pm(z) \quad (2)
\end{aligned}$$

for the $N=4$ SU(2)_k superconformal algebras, while the boundary conditions are im-

posed by use of one parameter η as

$$\begin{aligned} T^\pm(z) &= e^{\mp 2i\pi\eta} T^\pm(e^{2i\pi} z) , \\ G^\mp(z) &= -e^{\pm i\pi\eta} G^\mp(e^{2i\pi} z) , \\ \bar{G}_\pm(z) &= -e^{\mp i\pi\eta} \bar{G}_\pm(e^{2i\pi} z) . \end{aligned} \quad (3)$$

This η phase can be gauged away through the use of the local automorphism Eq.(2) by the choice $\alpha(z) = i\eta \log z$ [10].

For the convenience of our later use we shall give here the N=4 $SU(2)_k$ superconformal algebras in terms of Fourier components:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{k}{2}m(m^2-1)\delta_{m+n,0}, \\ [T_m^i, T_n^j] &= i\epsilon^{ijk}\eta_{kl}T_{m+n}^l + \frac{k}{2}m\delta_{m+n,0}, \quad [L_m, T_n^i] = -nT_{m+n}^i, \\ [T_m^i, G_r^a] &= -\frac{1}{2}(\sigma^i)^a_b G_{m+r}^b, \quad [T_m^i, \bar{G}_{a,r}] = \frac{1}{2}\bar{G}_{b,m+r}(\sigma^i)^b_a, \\ [L_m, G_r^a] &= (\frac{1}{2}m-r)G_{m+r}^a, \quad [L_m, \bar{G}_{a,r}] = (\frac{1}{2}m-r)\bar{G}_{a,m+r}, \\ \{G_r^a, G_s^b\} &= 0, \quad \{\bar{G}_{a,r}, \bar{G}_{b,s}\} = 0, \\ \{G_r^a, \bar{G}_{b,s}\} &= 2\delta^a_b L_{r+s} - 2(r-s)(\sigma^i)^a_b \eta_{ij} T_{r+s}^j + \frac{k}{2}(4r^2-1)\delta_{r+s,0}\delta^a_b \end{aligned} \quad (4)$$

where the following Fourier modings are to be given to complete the definition of the algebras: L_m, T_m^0 ($m \in Z$), T_m^\pm ($m \in Z \mp \eta$), $G_r^\pm, \bar{G}_{\pm,r}$ ($r \in Z \mp \frac{\eta}{2}$).

The equivalence of the algebras which differ by the value of the parameter η can be shown by expressing the generators of the algebras twisted by η in terms of the generators of the $\eta = 0$ (Ramond) algebra through the relations given by Eq.(2) with the choice of $\alpha(z) = i\eta \log z$:

$$\begin{aligned} L_R(z) &= L(z) - \frac{\eta}{z}T^0(z) + \frac{k\eta^2}{4z^2}, \\ T_R^0(z) &= T^0(z) - \frac{k\eta}{2z}, \quad T_R^\pm(z) = z^{\mp\eta}T^\pm(z), \\ G_R^\mp(z) &= z^{\pm\frac{\eta}{2}}G^\mp(z), \quad \bar{G}_{R\pm}(z) = z^{\mp\frac{\eta}{2}}\bar{G}_\pm(z). \end{aligned} \quad (5)$$

or by expressing the generators of the algebras twisted by $1-\eta$ in terms of the generators of the $\eta = 1$ (Neveu-Schwarz) algebra through the relations given by Eq.(2) with the choice of $\alpha(z) = -i(1-\eta) \log z$:

$$\begin{aligned} L_{\text{NS}}(z) &= L(z) + \frac{(1-\eta)}{z} T^0(z) + \frac{k(1-\eta)^2}{4z^2} , \\ T_{\text{NS}}^0(z) &= T^0(z) + \frac{k(1-\eta)}{2z} \quad T_{\text{NS}}^\pm(z) = z^{\pm(1-\eta)} T^\pm(z) , \\ G_{\text{NS}}^\mp(z) &= z^{\mp \frac{(1-\eta)}{2}} G^\mp(z) , \quad \bar{G}_{\text{NS}\pm}(z) = z^{\pm \frac{(1-\eta)}{2}} \bar{G}_\pm(z) . \end{aligned} \quad (6)$$

Since the algebra for each choice of η is equivalent to each other through the inner automorphism mentioned above, we basically consider in the following the typical choices of $\eta = 0$ and $\eta = 1$, and compare their resulting consequences. The value $\eta = 0$ in Eq.(3) corresponds to the Ramond (R) sector with integral r and s , while that of $\eta = 1$ to the Neveu-Schwarz (NS) one with half-integral values of them. The spectral flow between the NS and R sectors is obtained either by putting $\eta = 1$ in Eq.(5) or by putting $\eta = 0$ in Eq.(6).

For a general value of η we consider the *R-type* raising operators that are given as follows:

$$\begin{aligned} L_n & \quad (n > 0) , \\ T_{n-i\eta}^i & \quad (n > 0 \quad \text{or} \quad i = + \text{ and } n = 0) , \\ G_{n-\frac{1}{2}a\eta}^a & \quad (n > 0 \quad \text{or} \quad a = + \text{ and } n = 0) , \\ \bar{G}_{a,n-\frac{1}{2}a\eta} & \quad (n > 0 \quad \text{or} \quad a = + \text{ and } n = 0) , \end{aligned} \quad (7)$$

where $n \in Z$, and we have used the following notation: $i\eta = (\pm\eta, 0)$ for $i = (\pm, 0)$ and $a\eta = \pm\eta$ for $a = \pm$. The above choice corresponds to taking the raising operators among the generators of the Ramond ($\eta = 0$) sector to be the normal ones as are usually chosen, which can be obtained just by putting $\eta = 0$ in Eq.(7).

But we should note that for the NS sector with $\eta = 1$, the above choice of Eq.(7) amounts to taking the raising operators to be the *tilted* ones that are obtained by

putting $\eta = 1$ in Eq.(7) rather than the ordinary raising operators that are commonly used for the NS sector and are given by putting $\eta = 1$ in the following *NS-type* conditions:

$$\begin{aligned}
L_n & \quad (n > 0) , \\
T_{n+i(1-\eta)}^i & \quad (n > 0 \quad \text{or} \quad i = + \text{ and } n = 0) , \\
G_{n'+\frac{a(1-\eta)}{2}}^a & \quad (n' \geq \frac{1}{2}) , \\
\bar{G}_{a,n'+\frac{a(1-\eta)}{2}} & \quad (n' \geq \frac{1}{2}) ,
\end{aligned} \tag{8}$$

where $n \in Z$, $n' \in Z + \frac{1}{2}$. This implies that the hws's usually constructed for the NS sector by the condition of being killed by all the raising operators X_+ obtained by putting $\eta = 1$ in Eq.(8) do not correspond to the hws's of the R sector, but rather the latter should be reconstructed properly from the former if one started with the hws's of the NS sector. This procedure will be exemplified for some relevant cases later.

We take the Cartan subalgebra to be $\{L_0, T_0^0, k\}$, and the lowering operators to be the remaining generators. Then we define a highest weight representation (hwrep) of the algebras Eq.(4) to be one containing a highest weight state (hws) vector $|h, \ell\rangle$ such that

$$\begin{aligned}
L_0|h, \ell\rangle &= h|h, \ell\rangle , \\
T_0^0|h, \ell\rangle &= \ell|h, \ell\rangle ,
\end{aligned} \tag{9}$$

and

$$X_+|h, \ell\rangle = 0 , \tag{10}$$

for all raising operators X_+ .

Our FF representations of the $N = 4$ $SU(2)_k$ superconformal algebra are constructed in terms of four bosons $\varphi_\alpha(z)$ ($\alpha = 1, 2, 3, 4$) and four real fermions forming a pair of complex fermion doublet $\gamma^a(z)$ and antidoublet $\bar{\gamma}_a(z)$ ($a = 1, 2$ or \pm) under

$SU(2)_k$:

$$\begin{aligned} \left(\gamma^a(z) \right) &= \begin{pmatrix} \gamma^1(z) \\ \gamma^2(z) \end{pmatrix} = \begin{pmatrix} \gamma^-(z) \\ \gamma^+(z) \end{pmatrix} , \\ \left(\bar{\gamma}_a(z) \right) &= \left(\bar{\gamma}_1(z), \bar{\gamma}_2(z) \right) = \left(\bar{\gamma}_+(z), \bar{\gamma}_-(z) \right) , \end{aligned} \quad (11)$$

whose mode expansions are given by

$$\varphi_\alpha(z) = q_\alpha - ip_\alpha \log z + i \sum_{n \in Z, n \neq 0} \frac{\varphi_{\alpha,n}}{n} z^{-n} \quad (\alpha = 1, 2, 3, 4) , \quad (12)$$

and

$$\gamma^{\eta,a}(z) = \begin{cases} z^{-\frac{a(1-\eta)}{2}} \gamma_{\text{NS}}^a(z) \equiv \sum_{n' \in Z + \frac{1}{2}} \gamma_{n'}^a z^{-n' - \frac{a(1-\eta)}{2} - \frac{1}{2}} & \text{NS-type } (0 < \eta \leq 1) \\ z^{\frac{a\eta}{2}} \gamma_{\text{R}}^a(z) \equiv \sum_{n \in Z} \gamma_n^a z^{-n + \frac{a\eta}{2} - \frac{1}{2}} & \text{R-type } (0 \leq \eta < 1) , \end{cases} \quad (13)$$

$$\bar{\gamma}_a^\eta(z) = \begin{cases} z^{-\frac{a(1-\eta)}{2}} \bar{\gamma}_{\text{NS}a}(z) \equiv \sum_{n' \in Z + \frac{1}{2}} \bar{\gamma}_{a,n'} z^{-n' - \frac{a(1-\eta)}{2} - \frac{1}{2}} & \text{NS-type } (0 < \eta \leq 1) \\ z^{\frac{a\eta}{2}} \bar{\gamma}_{\text{Ra}}(z) \equiv \sum_{n \in Z} \bar{\gamma}_{a,n} z^{-n + \frac{a\eta}{2} - \frac{1}{2}} & \text{R-type } (0 \leq \eta < 1) . \end{cases} \quad (14)$$

In the following we mostly consider the R case ($\eta = 0$) only instead of the R-type case ($0 \leq \eta < 1$) for clarity and simplicity.

The commutators for the Fourier modes are

$$\begin{aligned} [\varphi_{\alpha,m}, \varphi_{\beta,n}] &= m \delta_{\alpha\beta} \delta_{m+n,0} , & [q_\alpha, p_\beta] &= i \delta_{\alpha\beta} , \\ [\varphi_{\alpha,n}, q_\beta] &= [\varphi_{\alpha,n}, p_\beta] = 0 , \end{aligned} \quad (15)$$

for the bosons, while for the fermions they are for the NS-type case ($m', n' \in Z + \frac{1}{2}$)

$$\begin{aligned} \{\gamma_{m'}^a, \bar{\gamma}_{b,n'}\} &= \delta^a_b \delta_{m'+n',0} = \delta_{a+b,0} \delta_{m'+n',0} , \\ \{\gamma_{m'}^a, \gamma_{n'}^b\} &= \{\bar{\gamma}_{a,m'}, \bar{\gamma}_{b,n'}\} = 0 , \end{aligned} \quad (16)$$

or for the R-type case ($m, n \in Z$)

$$\begin{aligned} \{\gamma_m^a, \bar{\gamma}_{b,n}\} &= \delta^a_b \delta_{m+n,0} = \delta_{a+b,0} \delta_{m+n,0} , \\ \{\gamma_m^a, \gamma_n^b\} &= \{\bar{\gamma}_{a,m}, \bar{\gamma}_{b,n}\} = 0 . \end{aligned} \quad (17)$$

The corresponding propagators for the boson fields are given by

$$\langle \varphi_\alpha(z) \partial \varphi_\beta(w) \rangle = \langle \partial \varphi_\beta(w) \varphi_\alpha(z) \rangle = \frac{\delta_{\alpha\beta}}{z-w} , \quad (18)$$

while those for the fermions are given by

$$\begin{aligned} \langle \bar{\gamma}_a^\eta(z) \gamma^{\eta,b}(w) \rangle &= -\langle \gamma^{\eta,b}(w) \bar{\gamma}_a^\eta(z) \rangle \\ &= \begin{cases} \frac{\delta_a^b}{z-w} \left(\frac{w}{z}\right)^{\frac{a(1-\eta)}{2}} & \text{for NS-type } (0 < \eta \leq 1) \\ \frac{\delta_a^b}{z-w} \frac{z+w}{2\sqrt{zw}} & \text{for R } (\eta = 0) , \end{cases} \end{aligned} \quad (19)$$

where we have defined the *NS-type* vacuum $|0\rangle$ as

$$\begin{aligned} \varphi_{\alpha,n}|0\rangle &= p_\alpha|0\rangle = 0 & (n > 0) , \\ \gamma_{n'}^a|0\rangle &= \bar{\gamma}_{a,n'}|0\rangle = 0 & (n' \geq \frac{1}{2}) , \end{aligned} \quad (20)$$

while for the zero modes of the R sector ($\eta = 0$) we have used the following definition of normal-ordering [17, 18, 19, 20]:

$$\bar{\gamma}_{a,0} \gamma_0^b = {}^\circ \bar{\gamma}_{a,0} \gamma_{0^\circ}^b + \langle \bar{\gamma}_{a,0} \gamma_0^b \rangle , \quad (21)$$

where

$${}^\circ \bar{\gamma}_{a,0} \gamma_{0^\circ}^b \equiv \frac{1}{2} [\bar{\gamma}_{a,0}, \gamma_0^b] , \quad \langle \bar{\gamma}_{a,0} \gamma_0^b \rangle \equiv \frac{1}{2} \{ \bar{\gamma}_{a,0}, \gamma_0^b \} = \frac{1}{2} \delta_a^b . \quad (22)$$

As to the non-zero modes ${}^\circ \circ$ is defined to reduce to the usual definition $::$ of normal-ordering where creation operators stand to the left of annihilation operators with appropriate sign factors.

As will be shown later, our formalism with the NS-type vacuum $|0\rangle$ allows one to obtain the generic expression for the conformal weights which is valid top from the NS ($\eta = 1$) sector further down to the R ($\eta = 0$) sector, reproducing the right value $\frac{1}{16} \times 4 = \frac{1}{4}$ of the well-known additional constant for the Ramond conformal

weight at the point $\eta = 0$. Note, however, that the expressions Eq(19) for the fermion propagators are not continuously connected at $\eta = 0$. This is simply because the two definitions of the normal ordering for the NS-type sector and that for the R sector are discontinuous due to the presence of the zero modes in the latter. We believe that what we have presented is the best one can do for the generic treatment for the conformal weights.

Now the FF representations are given with the parameters

$$\tau \equiv \sqrt{\frac{2}{\hat{k} + 2}} , \quad k \equiv \hat{k} + 1 , \quad \kappa \equiv \frac{i}{2} k \tau . \quad (23)$$

as follows:

$SU(2)_k$ Kac-Moody currents

$$T^i(z) = \begin{cases} J^{\eta,i}(z) + \frac{1}{2} : \bar{\gamma}^\eta(z) \sigma^i \gamma^\eta(z) : - \delta_{i,0} \frac{(1-\eta)}{2z} \\ = z^{-i(1-\eta)} J^i(z) - \delta_{i,0} \frac{\hat{k}(1-\eta)}{2z} \\ + z^{-i(1-\eta)} \frac{1}{2} : [\bar{\gamma}(z) \sigma^i \gamma(z)]_{\text{NS}} : - \delta_{i,0} \frac{(1-\eta)}{2z} \\ = z^{-i(1-\eta)} T_{\text{NS}}^i(z) - \delta_{i,0} \frac{k(1-\eta)}{2z} & \text{NS-type } (0 < \eta \leq 1) \\ J^i(z) + \frac{1}{2} : [\bar{\gamma}(z) \sigma^i \gamma(z)]_{\text{R}} : \\ = T_{\text{R}}^i(z) & \text{R } (\eta = 0) , \end{cases} \quad (24)$$

where

$$J^{\eta,\pm}(z) = z^{\mp(1-\eta)} J^\pm(z) , \quad J^{\eta,0}(z) = J^0(z) - \frac{\hat{k}(1-\eta)}{2z} , \quad (25)$$

with [21]

$$\begin{aligned} J^\pm(z) &= : \frac{i}{\sqrt{2}} \left(\sqrt{\frac{\hat{k}+2}{2}} \partial \varphi_1(z) \pm i \sqrt{\frac{\hat{k}}{2}} \partial \varphi_2(z) \right) e^{\pm i \sqrt{\frac{2}{\hat{k}}} (\varphi_3(z) - i \varphi_2(z))} : , \\ J^0(z) &= i \sqrt{\frac{\hat{k}}{2}} \partial \varphi_3(z) . \end{aligned} \quad (26)$$

The boundary conditions Eq.(3) and the spectral flow Eq.(6) for $T^i(z)$ are obviously valid in Eq.(24) when we have

$$p_3 - ip_2 = 0 , \quad (27)$$

which is valid[16] for any conformal state in our FF representations, as will be clear from our later discussion on vertex operators.

Total energy-momentum tensor

$$L(z) = \left\{ \begin{array}{l} \frac{1}{\hat{k} + 2} \left[\sum_{i,j=\pm,0} : J^{\eta,i}(z) \eta_{ij} J^{\eta,j}(z) : \right] \\ \quad - \frac{1}{2} (\partial \varphi_4(z))^2 - i\kappa \partial^2 \varphi_4(z) \\ \quad + \frac{1}{2} : [\partial \bar{\gamma}^\eta(z) \cdot \gamma^\eta(z) - \bar{\gamma}^\eta(z) \cdot \partial \gamma^\eta(z)] : + \frac{(1-\eta)^2}{4z^2} \\ = -\frac{1}{2} \sum_{\alpha=1}^4 : (\partial \varphi_\alpha(z))^2 : + i\frac{\tau}{2} \partial^2 \varphi_1(z) - i\kappa \partial^2 \varphi_4(z) \\ \quad - \frac{1-\eta}{z} J^0(z) + \frac{\hat{k}(1-\eta)^2}{4z^2} \\ \quad + \frac{1}{2} : [\partial \bar{\gamma}(z) \cdot \gamma(z) - \bar{\gamma}(z) \cdot \partial \gamma(z)]_{\text{NS}} : \\ \quad - \frac{1-\eta}{z} \frac{1}{2} : [\bar{\gamma}(z) \sigma^0 \gamma(z)]_{\text{NS}} : + \frac{(1-\eta)^2}{4z^2} \\ = L_{\text{NS}}(z) - \frac{1-\eta}{z} T_{\text{NS}}^0(z) + \frac{k(1-\eta)^2}{4z^2} \quad \text{NS-type } (0 < \eta \leq 1) \\ \\ \frac{1}{\hat{k} + 2} \left[\sum_{i,j=\pm,0} : J^i(z) \eta_{ij} J^j(z) : \right] \\ \quad - \frac{1}{2} (\partial \varphi_4(z))^2 - i\kappa \partial^2 \varphi_4(z) \\ \quad + \frac{1}{2} \circ [\partial \bar{\gamma}(z) \cdot \gamma(z) - \bar{\gamma}(z) \cdot \partial \gamma(z)]_{\text{R}^\circ} \circ + \frac{1}{4z^2} \\ = -\frac{1}{2} \sum_{\alpha=1}^4 : (\partial \varphi_\alpha(z))^2 : + i\frac{\tau}{2} \partial^2 \varphi_1(z) - i\kappa \partial^2 \varphi_4(z) \\ \quad + \frac{1}{2} \circ [\partial \bar{\gamma}(z) \cdot \gamma(z) - \bar{\gamma}(z) \cdot \partial \gamma(z)]_{\text{R}^\circ} \circ + \frac{1}{4z^2} \\ = L_{\text{R}}(z) \quad \text{R } (\eta = 0) . \end{array} \right. \quad (28)$$

where the spectral flow Eq.(6) for $L(z)$ is found to hold by use of Eq.(24).

$N=4$ supercurrents

$$\begin{aligned}
G^a(z) = & \left\{ \begin{aligned} & \gamma^{\eta,a}(z) i \partial \varphi_4(z) - 2\kappa \partial \gamma^{\eta,a}(z) \\ & - i\tau J^{\eta,i}(z) \eta_{ij} \left(\sigma^j \gamma^\eta(z) \right)^a \\ & + i\tau \left[: \left(\bar{\gamma}^\eta(z) \cdot \gamma^\eta(z) \right) \gamma^{\eta,a}(z) : - \frac{a(1-\eta)}{2z} \gamma^{\eta,a}(z) \right] \\ & = z^{-\frac{a(1-\eta)}{2}} \left[\gamma^a(z) i \partial \varphi_4(z) - 2\kappa \partial \gamma^a(z) \right. \\ & \quad - i\tau J^i(z) \eta_{ij} \left(\sigma^j \gamma(z) \right)^a \\ & \quad \left. + i\tau : \left(\bar{\gamma}(z) \cdot \gamma(z) \right) \gamma^a(z) : \right]_{\text{NS}} \\ & = z^{-\frac{a(1-\eta)}{2}} G_{\text{NS}}^a(z) \end{aligned} \right. \quad \text{NS-type } (0 < \eta \leq 1) \\
& \left\{ \begin{aligned} & \left[\gamma^a(z) i \partial \varphi_4(z) - 2\kappa \partial \gamma^a(z) - i\tau J^i(z) \eta_{ij} \left(\sigma^j \gamma(z) \right)^a \right. \\ & \quad \left. + i\tau^\circ \left(\bar{\gamma}(z) \cdot \gamma(z) \right) \gamma^a(z)^\circ \right]_{\text{R}} \\ & = G_{\text{R}}^a(z) \end{aligned} \right. \quad \text{R } (\eta = 0) ,
\end{aligned} \tag{29}$$

$$\begin{aligned}
\bar{G}_a(z) = & \left\{ \begin{aligned}
& \bar{\gamma}_a^\eta(z) i \partial \varphi_4(z) - 2\kappa \partial \bar{\gamma}_a^\eta(z) \\
& + i\tau J^{\eta,i}(z) \eta_{ij} \left(\bar{\gamma}^\eta(z) \sigma^j \right)_a \\
& - i\tau \left[: \left(\bar{\gamma}^\eta(z) \cdot \gamma^\eta(z) \right) \bar{\gamma}_a^\eta(z) : + \frac{a(1-\eta)}{2z} \bar{\gamma}_a^\eta(z) \right] \\
& = z^{-\frac{a(1-\eta)}{2}} \left[\bar{\gamma}_a(z) i \partial \varphi_4(z) - 2\kappa \partial \bar{\gamma}_a(z) \right. \\
& \quad + i\tau J^i(z) \eta_{ij} \left(\bar{\gamma}(z) \sigma^j \right)_a \\
& \quad \left. - i\tau : \left(\bar{\gamma}(z) \cdot \gamma(z) \right) \bar{\gamma}_a(z) : \right]_{\text{NS}} \\
& = z^{-\frac{a(1-\eta)}{2}} \bar{G}_{\text{NS}a}(z) \qquad \text{NS-type } (0 < \eta \leq 1) \\
& \left[\bar{\gamma}_a(z) i \partial \varphi_4(z) - 2\kappa \partial \bar{\gamma}_a(z) + i\tau J^i(z) \eta_{ij} \left(\bar{\gamma}(z) \sigma^j \right)_a \right. \\
& \quad \left. - i\tau^\circ \left(\bar{\gamma}(z) \cdot \gamma(z) \right) \bar{\gamma}_a(z)^\circ \right]_{\text{R}} \\
& = \bar{G}_{\text{Ra}}(z) \qquad \text{R } (\eta = 0) .
\end{aligned} \right. \tag{30}
\end{aligned}$$

where the boundary conditions Eq.(3) and the spectral flow Eq.(6) for $G^a(z)$ and $\bar{G}_a(z)$ are explicitly seen to be valid.

Now we consider the following basic vertex operators[7, 8]

$$\begin{aligned}
V(t, j, j_0; z) &= : e^{it\varphi_4(z)} V_{j,j_0}(z) : , \\
V_{j,j_0}(z) &= : e^{-ij\tau\varphi_1(z)} e^{ij_0\sqrt{\frac{2}{k}}(\varphi_3(z) - i\varphi_2(z))} : . \tag{31}
\end{aligned}$$

Note that the following OPE relations hold[7, 8]:

$$\begin{aligned}
J^0(z) V_{j,j_0}(w) &\sim \frac{j_0}{z-w} V_{j,j_0}(w) , \\
\sqrt{2} J^\pm(z) V_{j,j_0}(w) &\sim \frac{-j \pm j_0}{z-w} V_{j,j_0 \pm 1}(w) . \tag{32}
\end{aligned}$$

A primary state with conformal dimension $h(t, j)$ and $\text{SU}(2)$ spin (j, j_0) in the NS sector is obtained by operating the vertex operator on the NS-type vacuum as

$$|h(t, j), j_0\rangle \sim V(t, j, j_0; z=0)|0\rangle , \tag{33}$$

where the conformal weight $h(t, j)$ in the NS sector is given by

$$\begin{aligned}
h(t, j) &\equiv \frac{t^2}{2} + \kappa t + \frac{\tau^2}{2} j(j+1) \\
&= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \left(j + \frac{1}{2}\right)^2 + \frac{\hat{k}}{4} \\
&= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \left(j - \frac{k}{2}\right)^2 + j .
\end{aligned} \tag{34}$$

The last identity plays a crucial role in the following discussions.

Now the primary state Eq.(33), being a hws vector in the NS sector when $j_0 = j$, also stands for a NS-type hws vector $|h_\eta, \ell_\eta\rangle$ for the generators given by Eqs.(24),(28),(29) and (30). To be more explicit, we have

$$\begin{aligned}
\left(L_0 = L_{\text{NS},0} - (1 - \eta) T_{\text{NS},0}^0 + \frac{k(1 - \eta)^2}{4} \right) |h_\eta, \ell_\eta\rangle &= h_\eta |h_\eta, \ell_\eta\rangle \\
\left(T_0^0 = T_{\text{NS},0}^0 - \frac{k(1 - \eta)}{2} \right) |h_\eta, \ell_\eta\rangle &= \ell_\eta |h_\eta, \ell_\eta\rangle ,
\end{aligned} \tag{35}$$

where the state $|h_\eta, \ell_\eta\rangle$ is written as

$$\begin{aligned}
&\left| h_\eta \equiv h(t, j) - (1 - \eta)j + \frac{k}{4}(1 - \eta)^2, \ell_\eta \equiv j - \frac{k}{2}(1 - \eta) \right\rangle \\
&\equiv |h(t, j), j_0 = j\rangle \sim V(t, j, j_0 = j; z = 0)|0\rangle .
\end{aligned} \tag{36}$$

Later on we find that $h_0 = h_{\text{R}}$, $\ell_0 = \ell_{\text{R}}$ and $h_1 = h_{\text{NS}}$, $\ell_1 = \ell_{\text{NS}}$, and that h_η can also be rewritten as it should be like

$$h_\eta = h_{\text{R}} + \eta \ell_{\text{R}} + \eta^2 \frac{k}{4} , \quad \ell_\eta = \ell_{\text{R}} + \eta \frac{k}{2} . \tag{37}$$

Next we consider the R sector where the ground state vacua $|\pm, \pm; 0\rangle_{\text{R}}$ are quadruply degenerate and are defined by

$$\begin{aligned}
\varphi_{\alpha,n}|+, +; 0\rangle_{\text{R}} &= p_\alpha|+, +; 0\rangle_{\text{R}} = 0 & (n > 0) , \\
\gamma_n^a|+, +; 0\rangle_{\text{R}} &= \bar{\gamma}_{a,n}|+, +; 0\rangle_{\text{R}} = 0 & (n > 0 \quad \text{or} \quad a = + \text{ and } n = 0) ,
\end{aligned} \tag{38}$$

and

$$\begin{aligned} |-, +; 0\rangle_{\text{R}} &\equiv \gamma_0^- |+, +; 0\rangle_{\text{R}}, \quad |+, -; 0\rangle_{\text{R}} \equiv \bar{\gamma}_{-,0} |+, +; 0\rangle_{\text{R}}, \\ |-, -; 0\rangle_{\text{R}} &\equiv \bar{\gamma}_{-,0} \gamma_0^- |+, +; 0\rangle_{\text{R}} = \bar{\gamma}_{-,0} |-, +; 0\rangle_{\text{R}} = -\gamma_0^- |+, -; 0\rangle_{\text{R}}. \end{aligned} \quad (39)$$

Let us also note that

$$T_{\text{R},0}^0 |\pm, \pm; 0\rangle_{\text{R}} = \pm \frac{1}{2} |\pm, \pm; 0\rangle_{\text{R}}, \quad T_{\text{R},0}^{\pm} |\pm, \pm; 0\rangle_{\text{R}} = 0, \quad T_{\text{R},0}^{\mp} |\pm, \pm; 0\rangle_{\text{R}} = \frac{1}{\sqrt{2}} |\mp, \mp; 0\rangle_{\text{R}}, \quad (40)$$

and

$$T_{\text{R},0}^0 |\mp, \pm; 0\rangle_{\text{R}} = 0, \quad T_{\text{R},0}^{\pm} |\mp, \pm; 0\rangle_{\text{R}} = T_{\text{R},0}^{\mp} |\mp, \pm; 0\rangle_{\text{R}} = 0. \quad (41)$$

Therefore we have an iso-doublet $|\pm, \pm; 0\rangle_{\text{R}}$ and two iso-singlets $|\pm, \mp; 0\rangle_{\text{R}}$.

We find that $|+, +; 0\rangle_{\text{R}}$ is the highest weight Ramond (hw R) vacuum which satisfies the hws conditions of being annihilated by all the raising operators given by Eq.(7) with $\eta = 0$. Also we note that

$$L_{\text{R},0} |+, +; 0\rangle_{\text{R}} = \frac{1}{4} |+, +; 0\rangle_{\text{R}}, \quad T_{\text{R},0}^0 |+, +; 0\rangle_{\text{R}} = \frac{1}{2} |+, +; 0\rangle_{\text{R}}. \quad (42)$$

A hws vector $|h, \ell\rangle_{\text{R}}$ in the Ramond sector can be constructed as follows. We first apply the vertex operator on the hw R vacuum $|+, +; 0\rangle_{\text{R}}$ to obtain a primary state in the R sector as

$$\left| h(t, j), j_0; \lambda = +\frac{1}{2} \right\rangle_{\text{R}} \sim V(t, j, j_0; z=0) |+, +; 0\rangle_{\text{R}}. \quad (43)$$

where $\lambda = \left(\pm\frac{1}{2}, 0^{\pm}\right)$ generically denotes the $T_{\text{R},0}^0$ eigenvalues of the ground state vacua $|\pm, \pm; 0\rangle_{\text{R}}$ and $|\mp, \pm; 0\rangle_{\text{R}}$. Here we define for later use the primary states $|h(t, j), j_0; \lambda\rangle_{\text{R}}$ with other values of λ as

$$\begin{aligned} \left| h(t, j), j_0; \lambda = 0^+ \right\rangle_{\text{R}} &\equiv G_{\text{R},0}^- \left| h(t, j), j_0; \lambda = +\frac{1}{2} \right\rangle_{\text{R}}, \\ \left| h(t, j), j_0; \lambda = 0^- \right\rangle_{\text{R}} &\equiv \bar{G}_{\text{R},0} \left| h(t, j), j_0; \lambda = +\frac{1}{2} \right\rangle_{\text{R}}, \end{aligned}$$

$$\begin{aligned}
\left| h(t, j), j_0; \lambda = -\frac{1}{2} \right\rangle_{\text{R}} &\equiv \bar{G}_{\text{R},0} G_{\text{R},0}^- \left| h(t, j), j_0; \lambda = +\frac{1}{2} \right\rangle_{\text{R}} , \\
&\equiv -G_{\text{R},0}^- \bar{G}_{\text{R},0} \left| h(t, j), j_0; \lambda = +\frac{1}{2} \right\rangle_{\text{R}} .
\end{aligned} \tag{44}$$

Then we get a hws vector $|h, \ell\rangle_{\text{R}}$ by putting $j_0 = j$ in Eq.(43):

$$L_{\text{R},0} |h_{\text{R}}, \ell_{\text{R}}\rangle_{\text{R}} = h_{\text{R}} |h_{\text{R}}, \ell_{\text{R}}\rangle_{\text{R}} , \quad T_{\text{R},0}^0 |h_{\text{R}}, \ell_{\text{R}}\rangle_{\text{R}} = \ell_{\text{R}} |h_{\text{R}}, \ell_{\text{R}}\rangle_{\text{R}} , \tag{45}$$

where

$$\begin{aligned}
\left| h_{\text{R}} \equiv h(t, j) + \frac{1}{4}, \ell_{\text{R}} \equiv j + \frac{1}{2} \right\rangle_{\text{R}} &\equiv \left| h(t, j), j_0 = j; \lambda = +\frac{1}{2} \right\rangle_{\text{R}} \\
&\sim V(t, j, j_0 = j; z = 0) |+, +; 0\rangle_{\text{R}} .
\end{aligned} \tag{46}$$

From Eq.(34) we have

$$\begin{aligned}
h_{\text{R}} &= h(t, j) + \frac{1}{4} = \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \left(j + \frac{1}{2} \right)^2 + \frac{\hat{k}}{4} + \frac{1}{4} \\
&= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \ell_{\text{R}}^2 + \frac{k}{4} \quad \left(\ell_{\text{R}} = j + \frac{1}{2} \right) .
\end{aligned} \tag{47}$$

On the other hand we get the NS conformal weight h_{NS} from Eqs.(34) and (36) by putting $\eta = 1$ as

$$\begin{aligned}
h_{\text{NS}} \equiv h(t, j) &= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \left(j - \frac{k}{2} \right)^2 + j \\
&= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2} \left(\ell_{\text{NS}} - \frac{k}{2} \right)^2 + \ell_{\text{NS}} \quad (\ell_{\text{NS}} = j) .
\end{aligned} \tag{48}$$

Therefore the relation

$$h_{\text{R}} = h_{\text{NS}} - \ell_{\text{NS}} + \frac{k}{4} \tag{49}$$

holds when the equality

$$\ell_{\text{R}} = \pm \left(\ell_{\text{NS}} - \frac{k}{2} \right) \tag{50}$$

is valid. The sign ambiguity in Eq.(50) will be clarified shortly, but here it is solved if one takes into account of the second relation in Eq.(37). Remarkably enough we thus

have found that the parameter space of our FF representations has the spectral flow built in consistently.

Here it is appropriate to make a remark on the tilted raising operators in the NS-type case. Suppose we start with the NS sector ($\eta = 1$) adopting the NS-type raising operators given by Eq.(8). The $SU(2)_k$ raising operators of $T_{n+i(1-\eta)}^i$ for example consist of the normal ones $\{T_0^+, T_1^\pm, T_2^\pm, \dots\}_{\text{NS}}$ at $\eta = 1$, whereas at $\eta = 0$ they provide the tilted ones $\{T_0^-, T_1^\pm, T_2^\pm, \dots\}_{\text{R}}$. Therefore, the NS hws vector with the highest eigenvalue of $\{T_0^0\}_{\text{NS}}$ at $\eta = 1$ turns into the *tilted* R hws vector with the lowest eigenvalue of the η -twisted $\{T_0^0\}_{\text{R}}$ at $\eta = 0$. The spectral flow top from the $\eta = 1$ point further down to the $\eta = 0$ point causes the reflection of the T_0^0 eigenvalues.

The reconstruction from the tilted R hws vector to provide the normal R hws vector is achieved by operating $\{T_{-1}^+\}_{\text{NS}}$ on the normal NS hws vector $(k - 2\ell_{\text{NS}})$ times as

$$\begin{aligned} & \left| h_{\text{R}} = h_{\text{NS}} - \ell_{\text{NS}} + \frac{k}{4}, \ell_{\text{R}} = \frac{k}{2} - \ell_{\text{NS}} \right\rangle \\ &= \left(T_{\text{NS}, -1}^+ \right)^{(k-2\ell_{\text{NS}})} \left| h_{\text{R}} = h_{\text{NS}} - \ell_{\text{NS}} + \frac{k}{4}, \ell_{\text{R}} = \ell_{\text{NS}} - \frac{k}{2} \right\rangle. \end{aligned} \quad (51)$$

where we have presumed that $(k - 2\ell_{\text{NS}})$ is a non-negative integer. This point will later be discussed in detail in connection with unitary representations. Eq.(51) also clarifies how the \pm sign in Eq.(50) comes about in the spectral flow.

Next we proceed to the discussion of unitarity conditions. First consider the following commutation relations

$$[T_{-m}^+, T_m^-] = T_0^0 - \frac{k}{2}m, \quad m = 1, 2, \dots \quad (52)$$

Sandwich them between the NS hws $|h_{\text{NS}} = h(t, j), \ell_{\text{NS}} = j\rangle$ or the R hws $|h_{\text{R}} = h(t, j) + \frac{1}{4}, \ell_{\text{R}} = j + \frac{1}{2}\rangle_{\text{R}}$, then we obtain $\ell_{\text{NS}} = j \leq \frac{k}{2}m$ or $\ell_{\text{R}} = j + \frac{1}{2} \leq \frac{k}{2}m$. We therefore get the constraints:

$$\text{NS} : \ell_{\text{NS}} = j \leq \frac{k}{2},$$

$$\text{R} : \ell_{\text{R}} = j + \frac{1}{2} \leq \frac{k}{2} . \quad (53)$$

Similarly sandwich the NS commutation relation $\{G_{\frac{1}{2}}^-, \bar{G}_{+,-\frac{1}{2}}\} = 2L_0 - 2T_0^0$ or the R commutation relation $\{G_0^-, \bar{G}_{+,0}\} = 2L_0 - \frac{k}{2}$ between the corresponding hws's, then we get

$$\begin{aligned} \text{NS} : h_{\text{NS}} &\geq \ell_{\text{NS}} , \\ \text{R} : h_{\text{R}} &\geq \frac{k}{4} . \end{aligned} \quad (54)$$

From $\text{SU}(2)$ theory we know that the unitarity condition generally requires ℓ_{NS} or ℓ_{R} to be non-negative intergers or half-integers. Consequently, unitary representations also require k to be non-negative integers: $k = 0, 1, 2, \dots$. Therefore we have from unitarity

$$\begin{aligned} \text{NS} : \ell_{\text{NS}} &= j = 0, \frac{1}{2}, 1, \dots, \frac{k}{2} , \\ \text{R} : \ell_{\text{R}} &= j + \frac{1}{2} = 0, \frac{1}{2}, 1, \dots, \frac{k}{2} . \end{aligned} \quad (55)$$

It is well known and can be seen from the Virasoro commutator with $m = -n = 1$ in Eq.(4) that the conformal weight $h(t, j)$ should be real and non-negative from unitarity. Suppose that we have a complex momentum

$$t = u + iv \quad (56)$$

for the φ_4 field. Then the conformal weights of the hws's for the NS and R secotors are expressed as

$$\begin{aligned} \text{NS} : h_{\text{NS}} &= h(u + iv, j) \\ &= \frac{u^2}{2} - \frac{1}{2} \left(v + \frac{k}{2} \tau \right)^2 - iu \left(v + \frac{k}{2} \tau \right) + \frac{\tau^2}{2} \left(j - \frac{k}{2} \right)^2 + j , \\ \text{R} : h_{\text{R}} &= h(u + iv, j) + \frac{1}{4} \\ &= \frac{u^2}{2} - \frac{1}{2} \left(v + \frac{k}{2} \tau \right)^2 - iu \left(v + \frac{k}{2} \tau \right) + \frac{\tau^2}{2} \left(j + \frac{1}{2} \right)^2 + \frac{k}{4} . \end{aligned} \quad (57)$$

where the identity in Eq.(34) has been fully used. The unitarity requires that

$$iu \left(v + \frac{k}{2} \tau \right) = 0, \quad \text{that is, either } u = 0 \text{ or } v + \frac{k}{2} \tau = 0, \quad (58)$$

and that, when $u = 0$ we have to have

$$\begin{aligned} \text{NS} : \quad & - \left(\frac{k}{2} - j \right) \tau \leq v + \frac{k}{2} \tau \leq \left(\frac{k}{2} - j \right) \tau, \\ \text{R} : \quad & - \left(j + \frac{1}{2} \right) \tau \leq v + \frac{k}{2} \tau \leq \left(j + \frac{1}{2} \right) \tau, \end{aligned} \quad (59)$$

from Eq.(54), while u can take any real value when $v + \frac{k}{2} \tau = 0$.

Therefore, the reality and the bounds Eq.(54) of the conformal weights for the NS and R sectors already provide the strong constraints on the allowed values taken by the complex momentum t . Fig.1 illustrates the allowed region of the values over which $t = u + iv$ sweeps over for the unitary representations in each sector.

As was already discussed by Eguchi and Taormina[14], and Yu[15], in contrast to the massive representations with $h_{\text{NS}} > \ell_{\text{NS}}$ or $h_{\text{R}} > \frac{k}{4}$, the massless representations with the lowest conformal weights $h_{\text{NS}} = \ell_{\text{NS}}$ or $h_{\text{R}} = \frac{k}{4}$ are special in that new singular vectors occur when $\ell_{\text{NS}} = j = \frac{k}{2}$ or $\ell_{\text{R}} = j + \frac{1}{2} = 0$. At those values of ℓ the non-negativeness of the Kac-determinants requires that the conformal weights are restricted to the lowest values possible given above. Otherwise the Kac-determinants become negative, thus the unitarity is violated. Cosequently, we have two classes of the unitary representations of the $N = 4$ $\text{SU}(2)_k$ algebra:

(A) *Massive representations*

$$\begin{cases} h_{\text{NS}} > \ell_{\text{NS}}, & \ell_{\text{NS}} = j = 0, \frac{1}{2}, 1, \dots, \frac{k-1}{2}, & \text{NS sector} \\ h_{\text{R}} > \frac{k}{4}, & \ell_{\text{R}} = j + \frac{1}{2} = \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{k}{2}, & \text{R sector} \end{cases} \quad (60)$$

(B) *Massless representations*

$$\begin{cases} h_{\text{NS}} = \ell_{\text{NS}} , & \ell_{\text{NS}} = j = 0, \frac{1}{2}, 1, \dots, \frac{k-1}{2}, \frac{k}{2} , & \text{NS sector} \\ h_{\text{R}} = \frac{k}{4} , & \ell_{\text{R}} = j + \frac{1}{2} = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{k}{2} , & \text{R sector} \end{cases} \quad (61)$$

As presented above, our simple arguments on the reality of the conformal weights expressed by the FF parameters have confirmed all these results, except the one that the conformal weights of the hws's in each sector are restricted to the massless values $h_{\text{NS}} = \ell_{\text{NS}}$ or $h_{\text{R}} = \frac{k}{4}$ when $\ell_{\text{NS}} = j = \frac{k}{2}$ or $\ell_{\text{R}} = j + \frac{1}{2} = 0$. The last result can only be drawn from the sub-determinant calculations. For the NS sector it was concluded by Yu[15] from the non-negativeness of the sub-determinants that when $\ell_{\text{NS}} = \frac{k}{2}$, the region $h_{\text{NS}} > \ell_{\text{NS}}$ is forbidden, thus the conformal weight of the hws being restricted to the value $h_{\text{NS}} = \ell_{\text{NS}}$. By use of the spectral flow of Eqs.(49) and (50), we can conclude that for the R sector the unitarity excludes the region $h_{\text{R}} > \frac{k}{4}$ when $\ell_{\text{R}} = 0$, thus restricting the conformal weight to the value $h_{\text{R}} = \frac{k}{4}$.

In conclusion we have incorporated the η phase of the spectral flow in the Feigin-Fuks representations of the $N = 4$ superconformal algebras through the twisted fermions and have explicitly demonstrated that the spectral flow is nicely operating among the generators of the FF representations. The validity of the spectral flow has also been investigated through the study of the FF parameters and the conditions for the unitary representations have been obtained in the FF parameter space from the reality of the conformal weights

whose expressions are given in terms of the FF parameters.

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Figure Caption

Fig.1

Complex $t = u + iv$ sweeps over the values on the thick lines for the unitary representations in the NS or R sector.

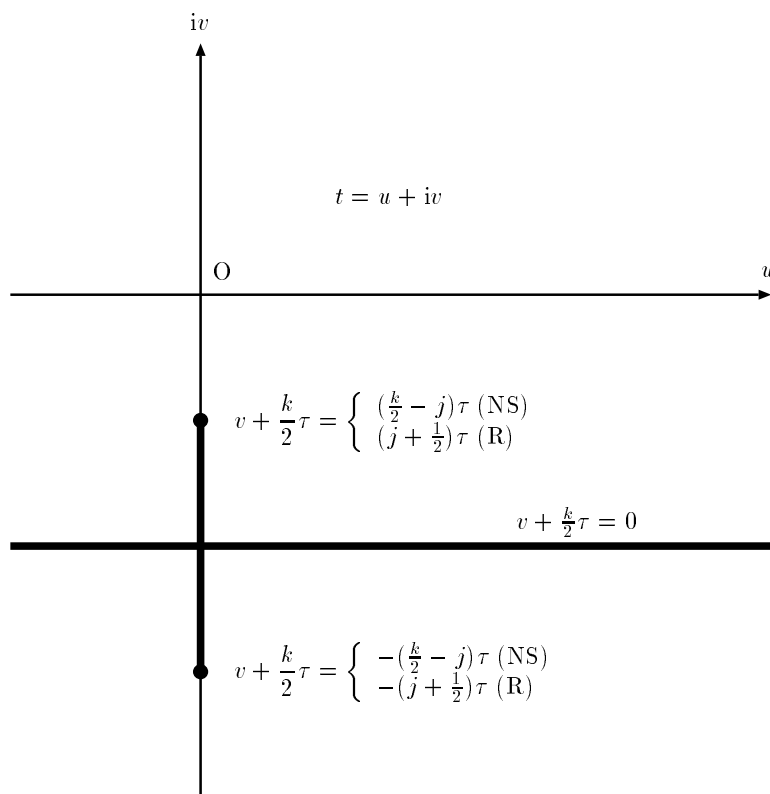


Fig.1